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On the numerical computation of the minimum-drag profile in laminar flow

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An approximation to the profile of given area with smallest drag in laminar flow is obtained (for Reynolds numbers between 10^3 and 10^5). It was shown previously by Pironneau (1974) that the skin friction on such a profile has to satisfy certain optimality conditions; the method used is based on these results. It was found that the optimum profile is long and thin (thickness-to-chord ratio about 10 %), the front end being shaped like a wedge of angle 90° and the rear end like a cusp. The drag is very close to the drag on a flat plate of equal length.

In a recent paper, Pironneau (1974) found that, if the flow around an aerofoil is governed by the steady Navier–Stokes equations

$$\nu\nabla^2\mathbf{u}-\mathbf{u}.\nabla\mathbf{u} = \nabla p, \quad \nabla.\mathbf{u} = 0, \quad \mathbf{u}|_S = 0, \quad \mathbf{u}|_\infty = \mathbf{u}_0, \tag{1}$$

the change in drag δF due to a hump of height $\alpha(s)$ on $S = \{\zeta(s) \mid s \in [0, L]\}$ (s = arc length on S) is given by

$$\delta F = \frac{\rho\nu}{u_0}\int_0^L \alpha(s)\frac{\partial\mathbf{u}}{\partial n}.\left(\frac{\partial\mathbf{u}}{\partial n}+2\frac{\partial\mathbf{w}}{\partial n}\right)ds + o(\alpha), \tag{2}$$

where $\mathbf{w}$, the 'co-state' of $\mathbf{u}$, is the solution of

$$\nu\nabla^2\mathbf{w}+\mathbf{u}.\nabla\mathbf{w}-\nabla\mathbf{u}.\mathbf{w} = \nabla q, \quad \nabla.\mathbf{w} = 0, \quad \mathbf{w}|_S = 0, \quad \mathbf{w}|_\infty = 0. \tag{3}$$

It was also shown that, given its area, the profile of smallest drag must be such that

$$\frac{\partial\mathbf{u}}{\partial n}.\left(\frac{\partial\mathbf{u}}{\partial n}+2\frac{\partial\mathbf{w}}{\partial n}\right) = \text{constant on } S. \tag{4}$$

Equation (2) implies that if (4) is not satisfied one can construct a profile of equal area but smaller drag by displacing S normally through a distance

$$\alpha(s) = -\lambda\left[\frac{\partial\mathbf{u}}{\partial n}.\left(\frac{\partial\mathbf{u}}{\partial n}+2\frac{\partial\mathbf{w}}{\partial n}\right)-k\right], \tag{5}$$

where k is the mean value of

$$\frac{\partial\mathbf{u}}{\partial n}.\left(\frac{\partial\mathbf{u}}{\partial n}+2\frac{\partial\mathbf{w}}{\partial n}\right)$$

on S and λ is small and positive.

It was also shown by Pironneau (1974) that $\mathbf{w}$ is equal to zero everywhere except in a boundary layer around the aerofoil and in its wake, and the boundary-layer equations for $\mathbf{w}$ were derived. From them it is easy to show that $\mathbf{Q} = 2\mathbf{w}+\mathbf{u}$ and $r = 2q+p-\frac{1}{2}\mathbf{u}\,.\,\mathbf{u}$ together satisfy

$$\left.\begin{array}{c} \nu\dfrac{\partial^2 Q_s}{\partial n^2}+u_s\dfrac{\partial Q_s}{\partial s}+u_n\dfrac{\partial Q_s}{\partial n}-\dfrac{\partial u_s}{\partial s}Q_s=\dfrac{\partial r}{\partial s}, \\ -\dfrac{\partial u_s}{\partial n}Q_s=\dfrac{\partial r}{\partial n}, \quad \nabla.\mathbf{Q}=0, \\ r|_C=0, \quad Q_s|_C=u_s, \\ Q|_S=0 \quad (\text{or } \partial Q_s/\partial n=0 \text{ on the axis of the wake of } S), \quad \mathbf{Q}(\infty,n)=\mathbf{u}_0, \end{array}\right\} \tag{6}$$

where Q_s is the boundary-layer value of Q and C is the outer envelope of the boundary layer. A Falkner–Skan type of change of variable shows that these equations have a parabolic character (for decreasing s) and that they are 'local' at $s = 0$; from this it was deduced in Pironneau (1974) that the front end of S must be tangential to a wedge of angle 90° (or a cone of angle 111° in three-dimensional axisymmetric flow). By making use of results in Landau & Liftshitz (1959, §4) and a local expansion of $\mathbf{w}$ it can be shown that if the boundary layer separates at $s = s_d$ then

$$w_s = a'n(s_d-s)^{\frac{1}{2}}+o(n)+o((s_d-s)^{\frac{1}{2}}).$$

Therefore (4) cannot be satisfied at $s = s_d$, which implies that the rear end of S must be shaped like a cusp. Thus we proceeded as follows.

We chose an initial guess S_0, symmetric and of area $a = 0{\cdot}147$, by specifying the outer boundary C_0 of its boundary layer. Then we computed in the potential region the tangential speed $U(s)$ on C_0 by the method of singularities (which reduces after discretization to the inversion of an $N\times N$ matrix, N being the number of points of discretization of C_0; see Luu 1971). Then u was computed in the boundary layer by integration of Prandtl's equations. We chose the following discretization (inspired by Keller 1974):

$$\left.\begin{array}{l} -(u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1})/(\Delta n)^2+\frac{1}{2}[(u_i^{j+1})^2 \\ \quad -(u_i^j)^2]/(s_{j+1}-s_j)+v_i^{j+1}(u_{i+1}^{j+1}-u_{i-1}^{j+1})/2\Delta n \\ \quad = \frac{1}{2}[(U^{j+1})^2-(U^j)^2]/(s_{j+1}-s_j), \\ v_i^{j+1}=v_{i-1}^{j+1}-\Delta n(u_i^{j+1}-u_i^j)/(s_{j+1}-s_j) \end{array}\right\} \quad (i=2,\ldots,M-1, \quad j\geqslant j_0), \tag{7}$$

with the boundary conditions

$$v_1^j=0, \quad u_M^j=U^j, \quad u_1^j=0 \quad \text{if} \quad s_j\leqslant \tfrac{1}{2}L,$$
$$u_1^j=u_2^j \quad \text{if} \quad s_j>\tfrac{1}{2}L, \quad u_i^{j_0}=u_d^i \quad (i=1,\ldots),$$

with $\Delta n = 0{\cdot}035$, $M = 250$, 47 sections on each half of S_0 and 11 sections in its wake; u_d was taken from Rosenhead (1963, p. 237).

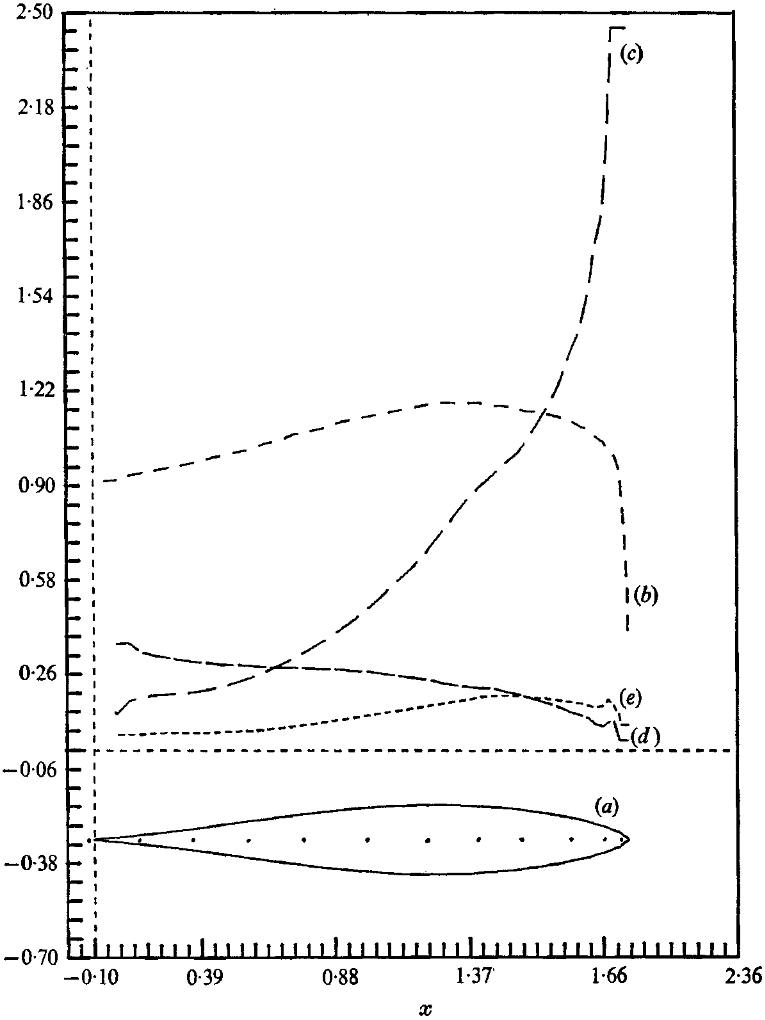

FIGURE 1. (*a*) Outer envelope C_3 of the boundary layer of S_3. (*b*) Tangential speed $U(s)$ on C_3. (*c*) Skin friction $\partial u_s/\partial n$. (*d*) Co-state function $\partial Q_s/\partial n$. (*e*) Gradient $(\partial u_s/\partial n) \times (\partial Q_s/\partial n)$.

Thus u^{j+1} was computed from u^j by solving the nonlinear system (7); for this, we used the method of over-relaxation (Cea & Glowinski 1973). Then Q^j was computed from Q^{j+1} in the boundary layer by solving

$$\left.\begin{aligned}(Q^j_{i-1} - 2Q^j_i + Q^j_{i+1})/\Delta n^2 + [Q^{j+1}_i/(s_{j+2} - s_{j+1}) - Q^j_i/s_{j+1} - s_j)]\, u^{j+1}_i \\ + (Q^j_{i+1} v^{j+1}_{i+1} - Q^j_{i-1} v^{j+1}_{i+1})/2\Delta n = r^{j+1}_i/(s_{j+2} - s_{j+1}) - r^j_i/(s_{j+1} - s_j), \\ - Q^j_i(u^{j+1}_{i+1} - u^{j+1}_{i-1})/2\Delta n = (r^j_{i+1} - r^j_i)/\Delta n,\end{aligned}\right\} \tag{8}$$

with the boundary conditions

$$Q^j_M = u^j_M, \quad r^j = 0, \quad Q^j_1 = 0 \ \text{ if } \ s \leqslant \tfrac{1}{2}L, \quad Q^j_1 = Q^j_2 \ \text{ if } \ s_j > \tfrac{1}{2}L, \quad Q^{58}_i = u^{58}_i.$$

Then $(\partial \mathbf{Q}/\partial n).(\partial \mathbf{u}/\partial n)$ and its mean were computed and C_1 obtained from C_0 by normal deformation of size α, according to (5) with $\lambda = 0{\cdot}04$.

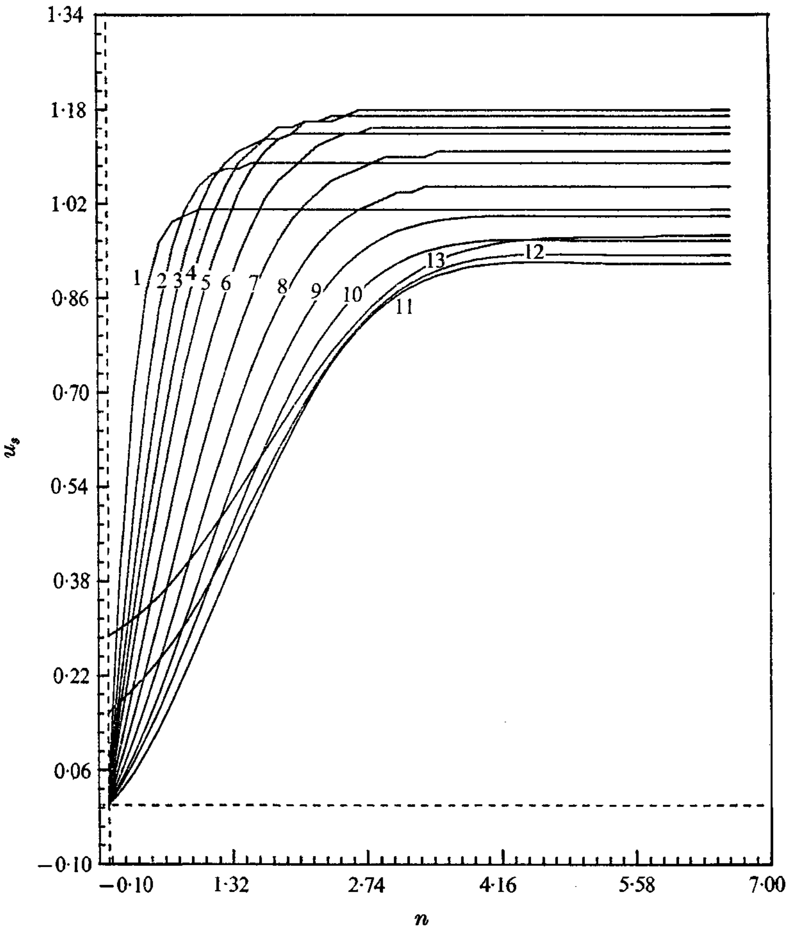

FIGURE 2. Tangential speed u_s in the boundary layer at the sections indicated by a dot on the axis of profile (*a*) in figure 1.

Each iteration of this method is quite costly (30 s on an IBM 370/168) and the precision is 10^{-3} for U and 10^{-2} for Q, which gives a precision of 0·05 for α/λ and 0·01 for F. For these reasons, we found that after three iterations it was not possible to improve C_3, for which the drag coefficient $C_D = 1{\cdot}33$ (to be compared with the value 1·328 for a flat plate), $k = 0{\cdot}10$ and the mean of

$$[(\partial \mathbf{u}/\partial n).(\partial \mathbf{Q}/\partial n)-k]^2$$

on S_3 is $0{\cdot}5\times10^{-2}$. Note that $(\partial \mathbf{u}/\partial n).(\partial \mathbf{Q}/\partial n)$ is fairly constant on the front of the profile while it is too small at the rear. We think that this is due to the discontinuity in $(\partial \mathbf{Q}/\partial s).(\partial \mathbf{u}/\partial s)$ at the cusp, which leads to numerical imprecision.

The profile S_3 is obtained from C_3 by subtracting the boundary-layer thickness (which is the only quantity that depends upon the Reynolds number). Other profiles of different area are obtained by expansion of C_3.

Thus, owing to the complexity of the problem, the precision obtained is not very good. However, this study shows that the method suggested in Pironneau

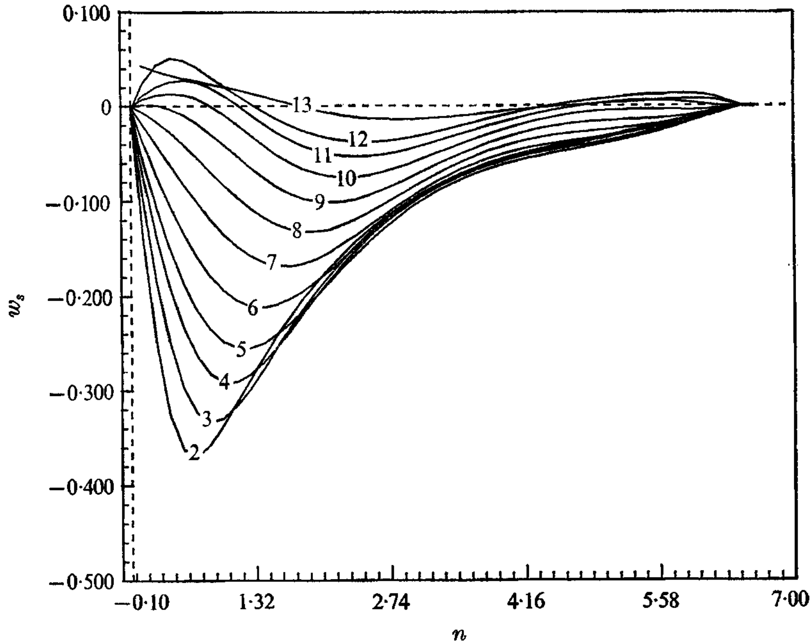

FIGURE 3. Tangential co-state w_s in the boundary layer at the sections indicated by a dot on the axis of profile (*a*) in figure 1.

(1974) works and that better precision can be obtained if one is ready to pay for it. Therefore a fluid mechanics laboratory wishing to solve similar problems of optimum design (such as the optimal-wing problem, for example) can proceed in this direction. On the other hand, we hope that, in the light of the results in figure 3, engineers will develop an intuitive feeling for the quantity

$$(\partial \mathbf{u}/\partial n).(\partial \mathbf{Q}/\partial n),$$

thereby reconciling this approach with optimum design and practical problems. The more important cases of turbulent boundary flow also wait for an interpretation of **Q** in order to be solved numerically.

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